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A Summation Formula Involving Fibonacci Digits

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A summation formula related to the Fibonacci expansion of integers is given.

I. INTRODUCTION

I.1. *A Result of Delange*

For every natural number n, $\sigma(n)$ denotes the sum of q-adic digits of n, where $q \geqslant 2$ is an integer. Generalizing a previous result of Trollope [3], Delange [1] proved:

THEOREM 1. *There exists a function F: $\mathbb{R} \to \mathbb{R}$, continuous, nowhere differentiable, of period* 1, *such that for any positive integer N,*

$$\sum_{n<N} \sigma(n) = \frac{q-1}{2 \operatorname{Log} q} N \operatorname{Log} N + NF\left(\frac{\operatorname{Log} N}{\operatorname{Log} q}\right).$$

Moreover, he gave the Fourier expansion of F. Our purpose here is to prove, in a different way, a similar formula concerning the sum of Fibonacci digits.

I.2. *Fibonacci Expansion*

The Fibonacci numbers are defined by $F_0 = F_1 = 1$ and

$$F_{k+2} = F_{k+1} + F_k \qquad \text{for every natural } k.$$

Every natural number n can be written

$$n = \sum_{k=1}^{\infty} a_k(n)\, F_k$$

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with $a_k(n) \in \{0, 1\}$ and $a_k(n)\, a_{k+1}(n) = 0$ for all $k \geqslant 1$, and the expansion is unique. From now on, $s(n)$ denotes the sum of Fibonacci digits of n:

$$s(n) = \sum_{k=1}^{\infty} a_k(n).$$

We set

$$S(N) = \sum_{n<N} s(n)$$

and for the sake of simplicity,

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \bar{\alpha} = \frac{1-\sqrt{5}}{2}$$

Our result is:

THEOREM 2. *There exists a function G: $\mathbb{R} \to \mathbb{R}$, continuous, nowhere differentiable of period* 1, *such that for any positive integer N,*

$$S(N) = \frac{3-\alpha}{5 \operatorname{Log} \alpha} N \operatorname{Log} N + NG\left(\frac{\operatorname{Log} N}{\operatorname{Log} \alpha}\right) + O(\operatorname{Log} N).$$

II. EXISTENCE AND CONTINUITY OF G

II.1. *Four Lemmas*

LEMMA 1. (1) *For all $k \in \mathbb{N}$, $F_k = ((5+\sqrt{5})/10)\, \alpha^k + ((5-\sqrt{5})/10)\, \bar{\alpha}^k$.*

(2) *For all $k \in \mathbb{N}$,*

$$S(F_k) = \left(\frac{k}{5} - \frac{\sqrt{5}}{25}\right) \alpha^k + \left(\frac{k}{5} + \frac{\sqrt{5}}{25}\right) \bar{\alpha}^k.$$

Proof. The first relation is well known. The second one is a consequence of

$$S(F_1) = 0, \qquad S(F_2) = 1,$$

and

$$\begin{aligned} S(F_{k+2}) &= S(F_{k+1}) + \sum_{n<F_k} s(n + F_{k+1}) \\ &= S(F_{k+1}) + S(F_k) + F_k \qquad \text{for all } k \in \mathbb{N}. \quad \blacksquare \end{aligned}$$

LEMMA 2. *If* $N=\sum_{h=1}^{\nu} \varepsilon_h F_h$ *with* $\varepsilon_\nu = 1$, *then*

$$\left| S(N) - \sum_{h=1}^{\nu} \varepsilon_h \alpha^h \left(\frac{h}{5} - \frac{\sqrt{5}}{25} + \frac{5+\sqrt{5}}{10} \sum_{h<k\leqslant\nu} \varepsilon_k \right) \right| = O(\operatorname{Log} N)$$

Proof.

$$\begin{aligned} S(N) &= \sum_{h=1}^{\nu} \sum_{m<\varepsilon_h F_h} s\left(m + \sum_{h<k\leqslant\nu} \varepsilon_k F_k \right) \\ &= \sum_{h=1}^{\nu} \varepsilon_h \left(\sum_{m<F_h} s(m) + F_h \sum_{h<k\leqslant\nu} \varepsilon_k \right). \end{aligned}$$

Lemma 2 is thus a consequence of Lemma 1 and of

$$\begin{aligned} &\left| \sum_{h=1}^{\nu} \varepsilon_h \bar{\alpha}^h \left(\frac{h}{5} + \frac{\sqrt{5}}{25} + \frac{5-\sqrt{5}}{10} \sum_{h<k\leqslant\nu} \varepsilon_k \right) \right| \\ &\quad\leqslant \sum_{h=1}^{\nu} |\bar{\alpha}|^h \left(\nu + \frac{\sqrt{5}}{25} \right) = O(\nu). \quad \blacksquare \end{aligned}$$

Now, we set $X=\sum_{h=1}^{\nu} \varepsilon_h \alpha^h$ and

$$S^*(X) = \sum_{h=1}^{\nu} \varepsilon_h \alpha^h \left(\frac{h}{5} - \frac{\sqrt{5}}{25} + \frac{5+\sqrt{5}}{10} \sum_{h<k\leqslant\nu} \varepsilon_k \right). \tag{1}$$

The following relations hold:

$$N = \frac{5+\sqrt{5}}{10} X + O(1) \tag{2}$$

and

$$S(N) = S^*(X) + O(\operatorname{Log} N) \qquad \text{from Lemma 2.} \tag{3}$$

LEMMA 3. *There exists a function* H: $\mathbb{R} \to \mathbb{R}$, *of period* 1, *such that*

$$S^*(X) = \frac{1}{5 \operatorname{Log} \alpha} X \operatorname{Log} X + X H\left(\frac{\operatorname{Log} X}{\operatorname{Log} \alpha} \right).$$

Proof. Let $\phi(X) = S^*(X) - ((X \operatorname{Log} X)/(5 \operatorname{Log} \alpha))$ from (1), we derive

$$\phi(\alpha X) - \alpha\phi(X) = S^*(\alpha X) - \alpha S^*(X) - \alpha X/5 = 0.$$

Therefore, $X \to X^{-1}\phi(X)$ is a function of $\operatorname{Log} X/\operatorname{Log} \alpha$ with period 1. $\blacksquare$

LEMMA 4. *Any real number y such that $1 \leqslant y < \alpha$ can be written*

$$y = \sum_{j=0}^{\infty} e_j \alpha^{-j} \qquad \text{with} \quad e_0 = 1,\ e_j \in \{0, 1\},\ e_j e_{j+1} = 0. \tag{4}$$

Moreover, this expansion is unique if we require that there is an infinity of indices j such that $e_j = e_{j+1} = 0$.

Proof. This result is proved in the book of Galambos [2]. ∎

If we put $y = \alpha^{-\nu} X$, it is plain that $1 \leqslant y < \alpha$. Thus, relation (4) holds. Moreover,

$$\begin{aligned} e_j &= 0 \qquad &\text{if} \quad j \geqslant \nu, \\ &= \varepsilon_{\nu - j} \qquad &\text{if} \quad j < \nu. \end{aligned} \tag{5}$$

LEMMA 5. *The function H is defined on the interval $[0, 1[$ by*

$$H\left(\frac{\operatorname{Log} y}{\operatorname{Log} \alpha}\right) = -\frac{\sqrt{5}}{25} - \frac{1}{5} \frac{\operatorname{Log} y}{\operatorname{Log} \alpha} + \frac{1}{y} \sum_{j=0}^{\infty} e_j \left(\frac{5+\sqrt{5}}{10} \sum_{r<j} e_r - \frac{j}{5} \right).$$

Proof.

$$\begin{aligned} H\left(\frac{\operatorname{Log} y}{\operatorname{Log} \alpha}\right) &= H\left(\frac{\operatorname{Log} X}{\operatorname{Log} \alpha}\right) = -\frac{\operatorname{Log} X}{5 \operatorname{Log} \alpha} + \frac{S^*(X)}{X} \\ &= -\frac{\operatorname{Log} y}{5 \operatorname{Log} \alpha} - \frac{\nu}{5} \\ &\quad + \frac{1}{y} \sum_{h=1}^{\nu} \varepsilon_h \alpha^{h-\nu} \left(\frac{h}{5} - \frac{\sqrt{5}}{25} + \frac{5+\sqrt{5}}{10} \sum_{h<k\leqslant \nu} \varepsilon_k \right) \text{ from (1)} \\ &= -\frac{\operatorname{Log} y}{5 \operatorname{Log} \alpha} - \frac{\nu}{5} \\ &\quad + \frac{1}{y} \sum_{j=0}^{\nu-1} e_j \alpha^{-j} \left(\frac{\nu - j}{5} - \frac{\sqrt{5}}{25} + \frac{5+\sqrt{5}}{10} \sum_{r<j} e_r \right) \text{ from (5)} \\ &= -\frac{\operatorname{Log} y}{5 \operatorname{Log} \alpha} \\ &\quad + \frac{1}{y} \sum_{j=0}^{\infty} e_j \alpha^{-j} \left(\frac{5+\sqrt{5}}{10} \sum_{r<j} e_r - \frac{j}{5} - \frac{\sqrt{5}}{25} \right). \end{aligned}$$

∎

II.2. *Continuity of H*

We only prove the continuity of H on $]0, 1[$ and leave to the reader the (similar) proof of

$$H(0) = -\frac{\sqrt{5}}{25} = \lim_{t \to 0^+} H(t) = \lim_{t \to 1^-} H(t).$$

Thus, we only have to show the continuity of

$$y \in [1, \alpha[\to \psi(y) = \sum_{j=0}^{\infty} e_j \alpha^{-j} \left(\frac{5+\sqrt{5}}{10} \sum_{r<j} e_r - \frac{j}{5} \right).$$

Let K denote a positive integer and let y and z satisfying

$$1 \leqslant y < z \leqslant \alpha \qquad \text{and} \qquad z - y < \alpha^{-K}. \tag{6}$$

With the expansion defined in (4), for y, we also expand z and write

$$z = \sum_{j=0}^{\infty} d_j \alpha^{-j}.$$

Let us put

$$p = \inf\{j; d_j \neq e_j\} \qquad \text{and} \qquad w = \sum_{j=0}^{p} d_j \alpha^{-j}.$$

Clearly, $d_p = 1$, $e_p = 0$, and $y < w \leqslant z$. From (6) $\sum_{j>p} d_j \alpha^{-j} = z - w < \alpha^{-K}$, thus we have

$$p \geqslant K$$

or (7)

$$p < K \text{ and } d_j = 0 \qquad \text{for} \quad p < j \leqslant K.$$

The inequality $w - y < \alpha^{-K}$ implies

$$\alpha^{-p} - \sum_{j>p} e_j \alpha^{-j} < \alpha^{-K}$$

hence

$$\sum_{r>0} e_{p+r} \alpha^{-r} > 1 - \alpha^{p-K}$$

and we get

$$e_{p+2k}=0,\ e_{p+2k+1}=1 \qquad \text{for} \quad 0\leqslant k<\left[\frac{K-p}{2}\right] \tag{8}$$

where $[u]=\operatorname{Max}\{m\in\mathbb{Z};\ m\leqslant u\}$. From (7), we deduce that

$$\begin{aligned}|\psi(z)-\psi(w)| &\leqslant \sum_{j>K} d_j\alpha^{-j}\left|\frac{5+\sqrt{5}}{10}\sum_{r<j} e_r-\frac{j}{5}\right| \\ &< \sum_{j>K} j\alpha^{-j}=O(K\alpha^{-K}).\end{aligned} \tag{9}$$

Since $d_j=e_j$ for $j<p$,

$$\begin{aligned}\psi(w)-\psi(y)=\alpha^{-p}&\left(\frac{5+\sqrt{5}}{10}\sum_{r<p} e_r-\frac{p}{5}\right) \\ &-\sum_{j>p} e_j\alpha^{-j}\left(\frac{5+\sqrt{5}}{10}\sum_{r<j} e_r-\frac{j}{5}\right).\end{aligned} \tag{10}$$

A straightforward calculation gives

$$\begin{aligned}&\alpha^{-p}\left(\frac{5+\sqrt{5}}{10}\sum_{r<p} e_r-\frac{p}{5}\right) \\ &\quad=\sum_{k=0}^{\infty}\alpha^{-p-1-2k}\left(\frac{5+\sqrt{5}}{10}\left(\sum_{r<p} e_r+k\right)-\frac{p+1+2k}{5}\right).\end{aligned} \tag{11}$$

(8), (10), and (11) imply

$$|\psi(w)-\psi(y)|<2\sum_{j\geqslant K-1} j\alpha^{-j}=O(K\alpha^{-K}). \tag{12}$$

(9) and (12) give the continuity of ψ. Moreover, with a good choice of K, we get

$$|\psi(z)-\psi(y)|=O(|z-y|\ \operatorname{Log}|z-y|).$$

From Lemma 5, H verifies

$$|H(t+h)-H(t)|=O(h\ \operatorname{Log}|h|), \tag{13}$$

where the constant in O is independent of t and h. ∎

II.3. *The function G*

From (2) and (13),

$$S^*(X)=\frac{1}{5\operatorname{Log}\alpha}\left(\frac{5-\sqrt{5}}{2}N+O(1)\right)\operatorname{Log}\left(\frac{5-\sqrt{5}}{2}N+O(1)\right)$$
$$+\left(\frac{5-\sqrt{5}}{2}N+O(1)\right)H\left(\frac{1}{\operatorname{Log}\alpha}\operatorname{Log}\left(\frac{5-\sqrt{5}}{2}N+O(1)\right)\right)$$
$$=\frac{3-\alpha}{5\operatorname{Log}\alpha}N\operatorname{Log}N+O(\operatorname{Log}N)+\frac{3-\alpha}{5\operatorname{Log}\alpha}\operatorname{Log}(3-\alpha)\cdot N+O(1)$$
$$+(3-\alpha)\,NH\left(\frac{\operatorname{Log}N}{\operatorname{Log}\alpha}+\frac{\operatorname{Log}(3-\alpha)}{\operatorname{Log}\alpha}+O\left(\frac{1}{N}\right)\right)$$
$$=\frac{3-\alpha}{5\operatorname{Log}\alpha}N\operatorname{Log}N+N\left(\frac{(3-\alpha)\operatorname{Log}(3-\alpha)}{5\operatorname{Log}\alpha}\right.$$
$$\left.+(3-\alpha)\,H\left(\frac{\operatorname{Log}N}{\operatorname{Log}\alpha}+\frac{\operatorname{Log}(3-\alpha)}{\operatorname{Log}\alpha}\right)\right)+O(\operatorname{Log}N).$$

Thus G is the continuous function of period 1 defined by

$$G(t)=\frac{(3-\alpha)\operatorname{Log}(3-\alpha)}{5\operatorname{Log}\alpha}+(3-\alpha)\;H\left(t+\frac{\operatorname{Log}(3-\alpha)}{\operatorname{Log}\alpha}\right).\quad \blacksquare$$

III. G Is Nowhere Differentiable

We have to show that ψ is nowhere differentiable. Let $y\in[1,\alpha[$. We put $y=\sum_{j=0}^{\infty}e_j\alpha^{-j}$ and for all $m\in\mathbb{N}^*$,

$$y_m=\sum_{j<m}e_j\alpha^{-j},\quad z_m=y_m+\alpha^{-1-m},\quad z'_m=y_m+\alpha^{-2-m}. \tag{14}$$

It is clear that

$$|y-y_m|\leqslant\alpha^{-m},\quad |y-z_m|\leqslant\alpha^{-m},\quad |y-z'_m|\leqslant\alpha^{-m}. \tag{15}$$

Moreover,

$$\psi(z_m)-\psi(y_m)=\alpha^{-1-m}\left(\frac{5+\sqrt{5}}{10}\sum_{j<m}e_j-\frac{m+1}{5}\right)$$
$$\psi(z'_m)-\psi(y_m)=\alpha^{-2-m}\left(\frac{5+\sqrt{5}}{10}\sum_{j<m}e_j-\frac{m+2}{5}\right)$$

so that

$$\alpha^2(\psi(z'_m) - \psi(y_m)) - \alpha(\psi(z_m) - \psi(y_m)) = -\frac{\alpha^{-m}}{5}. \tag{16}$$

If ψ was differentiable in y, with $\psi'(y) = a$, (14) and (15) would imply

$$|\alpha^2(\psi(z'_m) - \psi(y_m)) - a\alpha^{-m}| = o(\alpha^{-m}),$$
$$|\alpha(\psi(z_m) - \psi(y_m)) - a\alpha^{-m}| = o(\alpha^{-m}),$$

a contradiction to (16). G is nowhere differentiable.

References

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